New Pathways in Serial Isogons

Lee C. F. Sallows

If I have stood on the shoulders of giants it is because I tried to see further than they could.

—Isogones of Retsina (c. 666 BC)

A Pretty Polyomino

They say the road to Hell is paved with good intentions. A recent intention of mine was to solve a puzzle in taxicab geometry. During the attempt, absent-minded doodling on squared paper led to the incidental discovery of an arresting figure: a polyomino having eight sides of length 1, 2, . . . , 8 units, the latter occuring *in consecutive order* around the boundary (see Figure 1). This was already an interesting find. Yet, glancing again at my sketch the next day, I was seized by a wild surmise. A quick trial at once realised hope: the polyomino has a shape satisfying the Conway criterion [1], and is thus able to *pave the plane.* Now here was a prize to celebrate. As below, so above: The road to Heaven is paved with good in*v*entions.

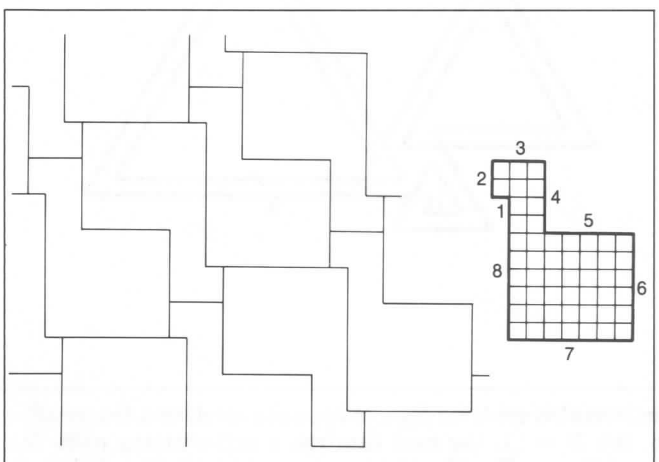
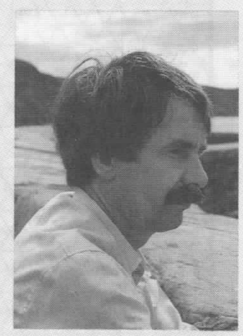


Figure 1. Tiling the plane with a serial-sided polyomino.

So I undertook further study of serial-sided polygons—or "golygons" as I playfully dubbed early specimens. Polygons may be defined to include *self-crossing* as well as simple figures, and so it is with golygons when defined as serial-sided, closed paths on a square



Lee C. F. Sallows

Raised in post-war London, Lee Sallows attended Dame Alice Owen's Grammar School For Boys, a posh establishment oozing with History and Tradition, where cane-wielding masters in flowing black gowns encouraged their erring flock toward the gilded spires of a *University.* Alas, our hero quickly departed unburdened by testimonials. However, at age 17, short-wave radio interests led to his gaining a ham operator's "ticket," callsign G3RGH, and experience won via design of transmitting and receiving gear proved a useful asset, since which time he has lived on his wits working in various corners of the electronics industry. In a belated bid to upstage Dame Alice, in 1971 he gained access to the University of Nijmegen in Holland, sneaking in via the back door as a member of their non-academic staff. Inspired by Martin Gardner, as a mathematical amateur Sallows has published a handful of articles, mostly on recreational topics. He enjoys an Erdös number of 2.

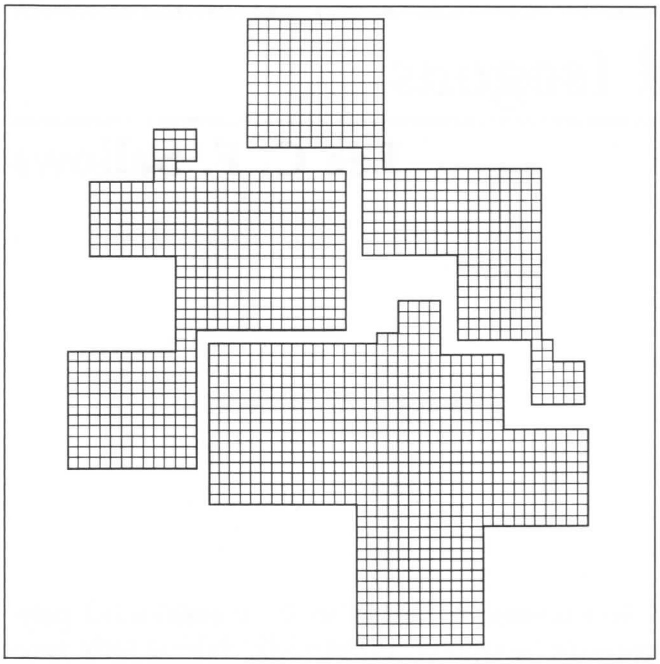


Figure 2. Three "golygons" of order 16. In all, there are 28 paths of order 16, the remaining 25 being self-crossing.

grid; *self-overlapping* line segments may occur also. It has been proved that the number of sides in these figures is always a multiple of eight. Figure 2 shows some examples. These polygons formed the focus of a memorable collaboration resulting in a joint article [2]. Interested readers may like to consult this paper, or a subsequent summary [3]. One let-down, however—among hundreds of cases discovered, Figure 1 remains the only instance with the paving property!

Beyond golygons, however, we have serial-sided isogons (*iso,* meaning equal; and *gon,* which means angled) in general. That is to say, closed serial-segment paths in which the absolute angle between consecutive segments (or sides, or edges) is again constant, but not necessarily 90 degrees. The term "absolute" stresses that angle *magnitudes* are equal; in zig-zag figures the sign of angles at different corners obviously varies. Thus, as with right-angled types, given the angle employed, any serial isogon is completely described by its sequence of left/right turns, as encountered in traversing the path in natural order of edges. Figure 3 shows some examples using angles of 60° and 120°, the earli-

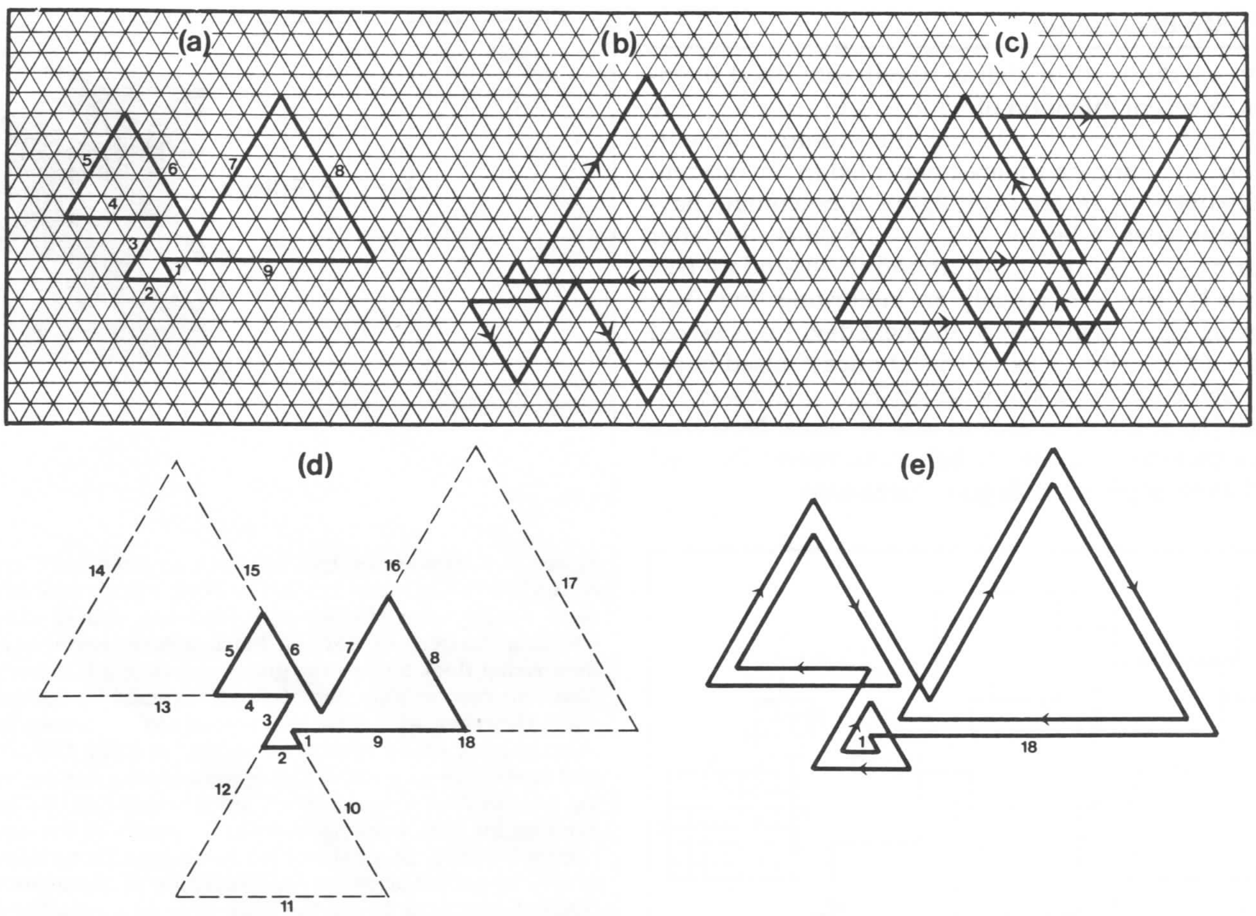


Figure 3. Serial isogons of 60° and 120°. These are all paths on an isometric grid; the figures opposite are drawn to a smaller scale. (a): $N = 9$, the shortest path for 60°, a simple polygon. (b): $N = 11$, the next shortest, a self-crossing path. (c): $N = 12$, one of the two self-crossing paths for this order. (d): Repeating a path with new segments of length $N + 1$, $N + 2$, . . . , produces a "second harmonic" (dotted) of the original (a). (e): The sequence of turns in path (a) is changed from

est specimens discovered. N is the *order* of the path, its number of edges.

Rational Isogons

For what angles can serial isogons be found? A full answer is still wanting, but an excellent start due to Hans Cornet is his proof that at least one such path exists for any angle α that is a rational fraction of 360 degrees, that is, for which $\alpha = (m/n) \cdot 360°$, m and n both positive integers. The detailed proof is on the long side, but at its heart is a simple recipe for constructing an isogon using any desired rational angle. The notions of edge direction and path turning angle are useful in explaining this.

Consider a moving point tracing an isogon in serial order of edges. By the *direction*, d_n, of an edge we mean that of the point tracing it, and by the *turning angle*, τ, of the path, we mean the angular deflection entailed in changing from one direction to the next. This is simply the (absolute) angle made between any edge and a line extending the previous edge, and is equal to $|180-\alpha|$ degrees (see Figure 4). Clearly, if α is rational then so is τ, implying that there exists an integer D such that $D \cdot \tau$ is a whole number. D is of course the denominator in the rational fraction τ, reduced to lowest terms. Thus, D repeated turns of τ degrees to right or to left equals some whole number of 360-degree rotations, meaning a return to the initial orientation. This shows that the number of available edge directions in any rational isogon cannot exceed D, and that they

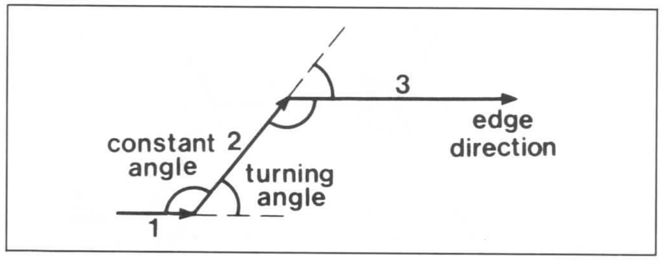
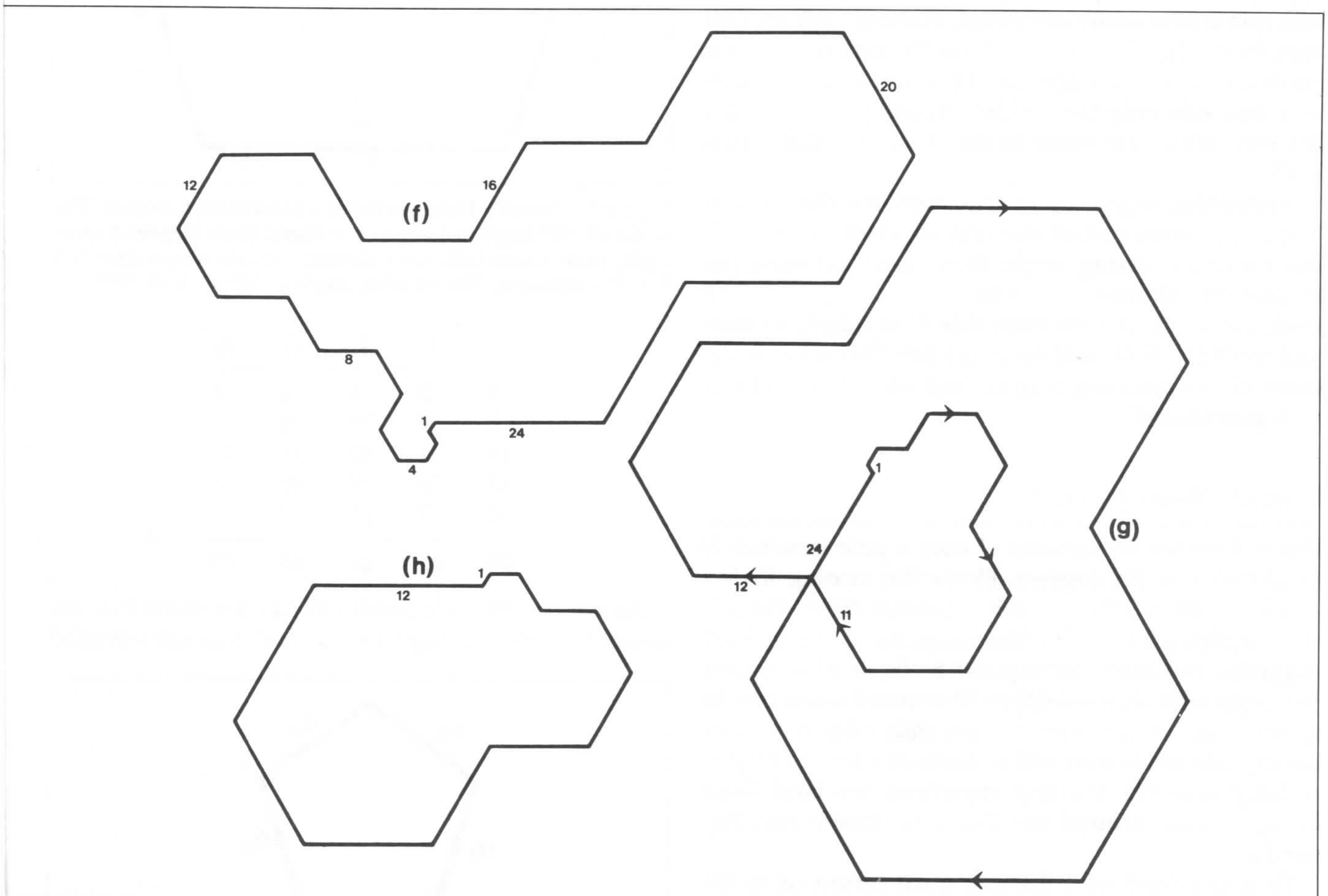


Figure 4. Every edge (1, 2, 3, . . .) points in a certain direction (d_1, d_2, d_3, . . .). The turning angle τ is the supplement of the constant angle α.



RRLRRLRRL to RRRRRLRRRRRLRRRRRL, a variation on Cornet's rule (see text). (f): N = 24, α = 120°, one of 20 simple paths from the total of 139 for this order. (g): N = 24, α = 120°; a self-crossing path. (h): N = 12, the shortest serial isogon for 120°. Note how the same figure is contained in (g).

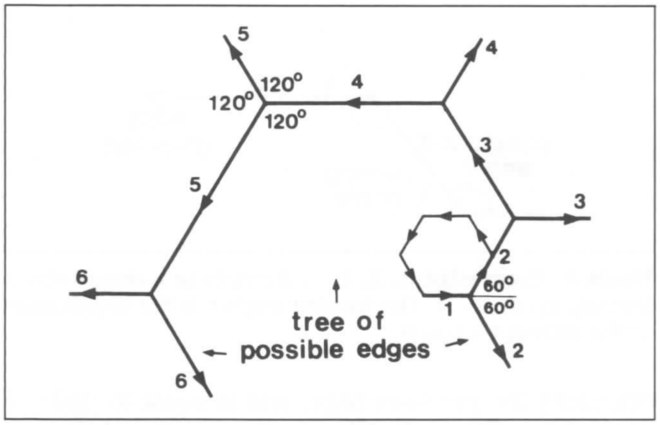


Figure 5. In a path using turning angle $\tau = 60° = \frac{1}{6} \cdot 360°$, all edge directions will correspond with the directed sides of a regular hexagon.

correlate with the sides of a regular D-gon, suitably oriented. In a path using $\alpha = 120$ degrees, for instance, $\tau = 60$ or $\frac{1}{6}$ of 360 degrees, so that $D = 6$, meaning that every edge parallels and points in the same direction as one of the directed sides of a regular hexagon, appropriately aligned (see Figure 5). Cornet's construction rule is now easily explained. Starting with the first turn from edge 1 to edge 2, form the rational angle α between successive edges so that every following turn is to one side only (say, right), *excepting* turns D, $2D$, $3D$, etc., which are made to the other side (left). That is all.

Appending segments in this way, we find that at length the open end of the first segment is rejoined, the resultant closing angle then made between the longest and shortest edges being α, as required. The final, automatic re-turn from side N to side 1, an integral multiple of D, will be to the left. But what is the order of the resulting isogon, and why is correct closure guaranteed?

Cornet's Proof (D Odd)

Figure 6 shows an instance of such a path in which $N = 25$ and $\alpha = 108$ degrees. Hence the turning angle $\tau = 180 - 108 = 72°$ or $\frac{1}{5}$ of 360 degrees, from which $D = 5$, implying 5 available directions. As the figure itself suggests, the latter correspond to those of a regular pentagon with directed edges all arrowed clockwise. In general, as already seen, the possible edge directions for any rational isogon will be those of a similar D-gon, suitably oriented. Starting anywhere, we label these $d_1, d_2, \ldots, d_D$, around the D-gon perimeter (see Figure 7).

To understand why the rule must return us to the starting point, consider a table showing how edges 1, 2, 3, . . . are allotted to directions d_1 to d_5 in Figure 6. Note the step left after every 4 steps right (d_5 is of course adjacent to d_1).

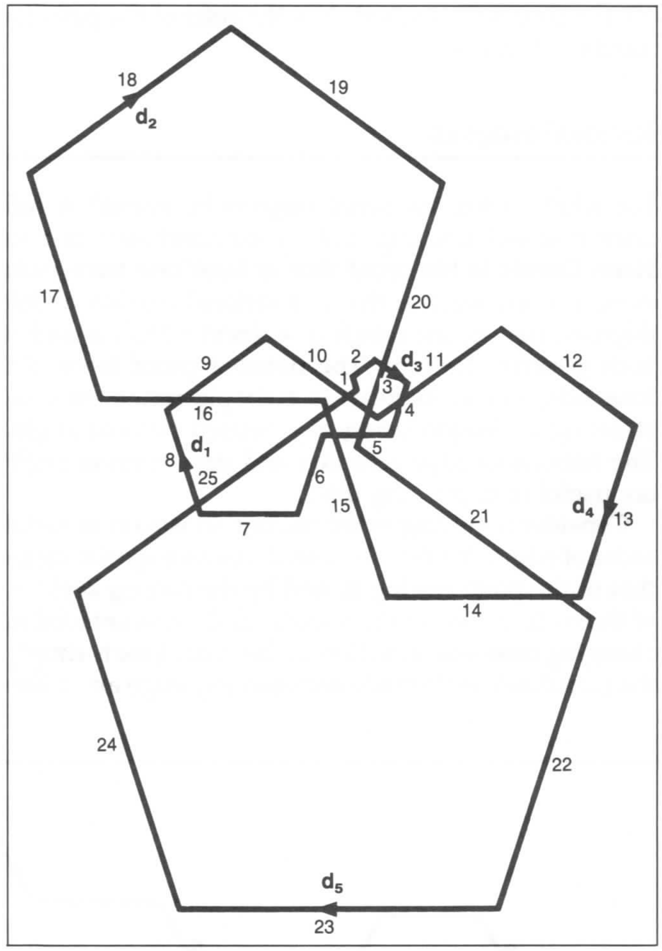


Figure 6. Cornet's Rule produces a serial-sided isogon. The angle of 108° implies 5 edge directions, thus: Repeat 4 turns right, then 1 turn left, until closure, which occurs after $5 \times 5 = 25$ segments. The turning angle is $72° = \frac{1}{5}$ of 360°.

d_1	d_2	d_3	d_4	d_5
1	2	3	4	5
8	9	10	6	7
15	11	12	13	14
17	18	19	20	16
24	25	21	22	23
65	65	65	65	65

+

As we see, the rule results in column sums that are equal. But this is to say that the total distance travelled

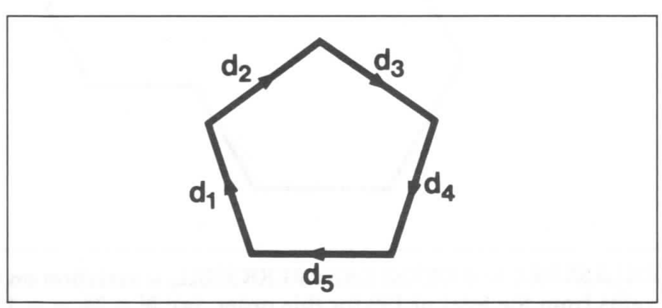


Figure 7. Starting anywhere, the directions of a regular pentagon are labelled d_1 to d_5.

in every direction is the same (65 units). The vectorial sum or final displacement from start to finish is thus the same in effect as traversing a regular pentagon of side 65; that is, zero displacement, indicating that the path ends where it begins.

Furthermore, edge 25, the longest, falls in a column adjacent to d_1; the angle formed with edge 1 is thus 108 degrees, as required. Since the table comprises D rows of D entries, in this case the order of the isogon is $D^2 = 25$.

It is easy to see why Cornet's rule must result in a table with these properties whenever D is odd. Consider the same table with $r \cdot D$ subtracted from every number in the rth row, row 0 being at the top.

d_1	d_2	d_3	d_4	d_5	
1	2	3	4	5	
3	4	5	1	2	
5	1	2	3	4	
2	3	4	5	1	
4	5	1	2	3	+
15	15	15	15	15	

A glance now shows that the columns produced by the rule are really cyclic permutations of the numbers 1, 2, . . . , D, added to which are the r terms $0D$, $1D$, $2D$, . . . , $(D - 1)D$, in every case. Hence column sums must always agree, their totals equalling $(\frac{1}{2}) \cdot D(D^2 + 1)$, as a simple calculation will show.

Likewise, the rows are also cyclic permutations of 1, 2, . . . , D. The table is thus a *latin square,* its bottom row being completed by an entry falling in column d_2. The correct closure angle is thus assured. In this light, Cornet's rule turns out to be not so different from one of those old-fashioned recipes for making a magic square!

In the above example τ was $\frac{1}{5}$ of 360°. Suppose, instead, the constant angle α had been 36°, so that τ becomes 144° or $\frac{2}{5}$ of 360°. Using arguments similar to the foregoing, it is easy to show that column sums, closure angle, and order all remain independent of the numerator in τ, provided the denominator (representing the number of directions, D) is unchanged if the new fraction is reduced to lowest terms. The following table illustrates our example. The corresponding isogon, a sorcerer's pentacle to delight any apprentice, is seen in Figure 8.

d_1	d_2	d_3	d_4	d_5	
1	4	2	5	3	
8	6	9	7	10	
15	13	11	14	12	
17	20	18	16	19	
24	22	25	23	21	+
65	65	65	65	65	

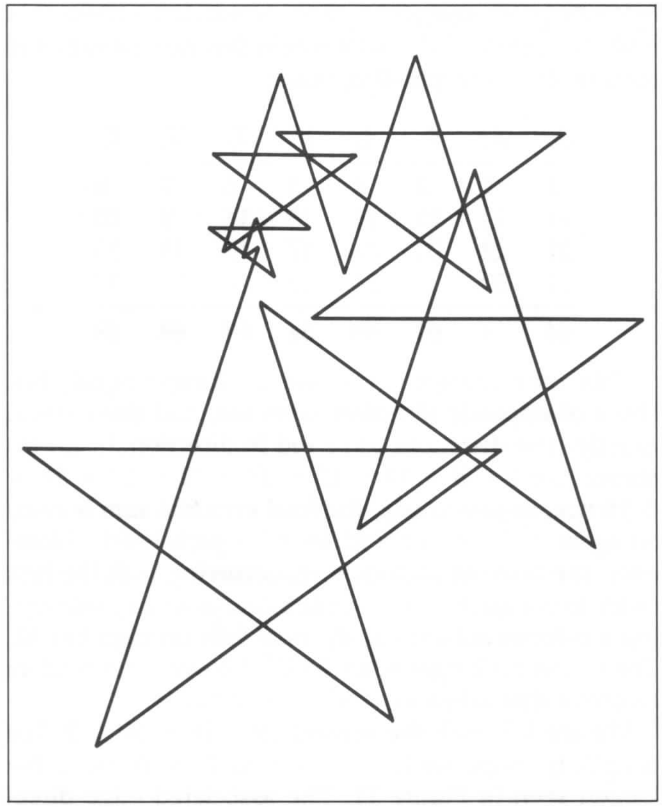


Figure 8. An isogon of order 25, using an angle of 36°. This is a close relative of Figure 6: the same rule applied with the turning angle now doubled to 144° = ⅖ of 360°.

Cornet's Proof (D Even)

What happens when the number of directions is even? Cornet distinguishes two cases, $D = 4k$, and $D = 4k + 2$; where k is a positive integer. Taking the first, suppose $k = 2$, so that $D = 8$, as in the isogon seen in Figure 9, where τ is 45 degrees, or ⅛ of 360°. The eight possible edge directions are then indicated by the sides of a regular octagon, as shown in Figure 10.

As the labelling reflects, here directions appear in

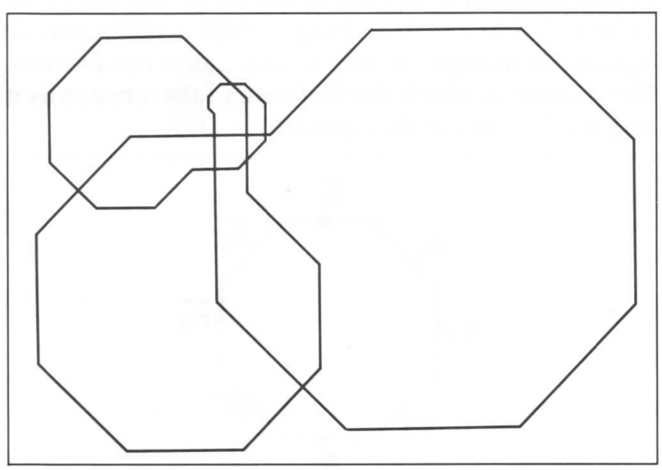


Figure 9. A path of order 32, with $\alpha = 135°$. The turning angle is 45° or ⅛ of 360°, so that D, the number of edge directions (8) is of form $D = 4k$, $k = 2$.

opposite pairs, something that cannot occur when D is odd. The effect of Cornet's rule is this new situation is seen in the corresponding table:

d_1	d_2	d_3	d_4	$\overline{d_1}$	$\overline{d_2}$	$\overline{d_3}$	$\overline{d_4}$	
1	2	3	4	5	6	7	8	
11	12	13	14	15	16	9	10	
21	22	23	24	17	18	19	20	
31	32	25	26	27	28	29	30	+
64	68	64	68	64	68	64	68	

This time column sums are no longer equal, but those of opposite direction pairs are, and thus cancel exactly. The distances traversed in direction d_1, for instance, are $1 - 5 + 11 - 15 - 17 + 21 - 27 + 31 = 0$. Hence, as previously, the final vectorial sum is zero, so again, path start coincides with path finish. However, the point of path closure, occurring with the first entry to complete a row while simultaneously occupying a column adjacent to d_1, now falls on segment 32. Thus, N is no longer equal to D^2. I leave it for readers to prove that when $D = 4k$, $N = D^2/2$.

We are left with the second case, $D = 4k + 2$. For simplicity, suppose $k = 1$, so that $D = 6$, as in the isogon seen in Figure 11. The associated edge directions are represented by the sides of a regular hexagon. D being even, directions again come in pairs of opposites. The corresponding table for Cornet's rule is then:

d_1	d_2	d_3	$\overline{d_1}$	$\overline{d_2}$	$\overline{d_3}$	
1	2	3	4	5	6	
9	10	11	12	7	8	
17	18	13	14	15	16	+
27	30	27	30	27	30	

Column sums are not equal, and neither are those of opposite direction pairs. However, the total displacement in *alternate* directions is the same: For d_1, d_3, and $\overline{d_2}$ it is 27–30, for d_2, $\overline{d_1}$, and $\overline{d_3}$ it is 30–27. The net effect is thus that of circumscribing a regular ($D/2$)-gon, or equilateral triangle of side 3, and again equals zero. The column in which the final entry falls remains as it was for $D = 4k$, so that again $N = D^2/2$.

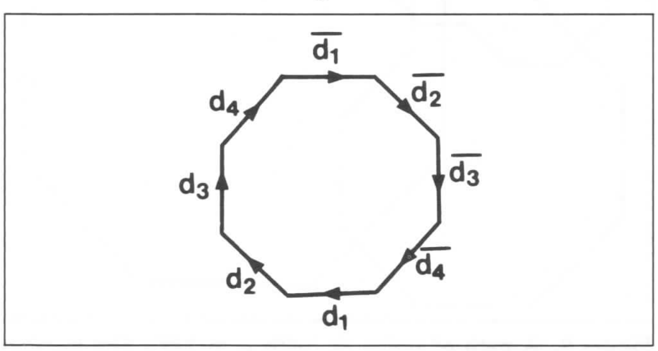


Figure 10. In a regular octagon the edge directions occur in opposite pairs.

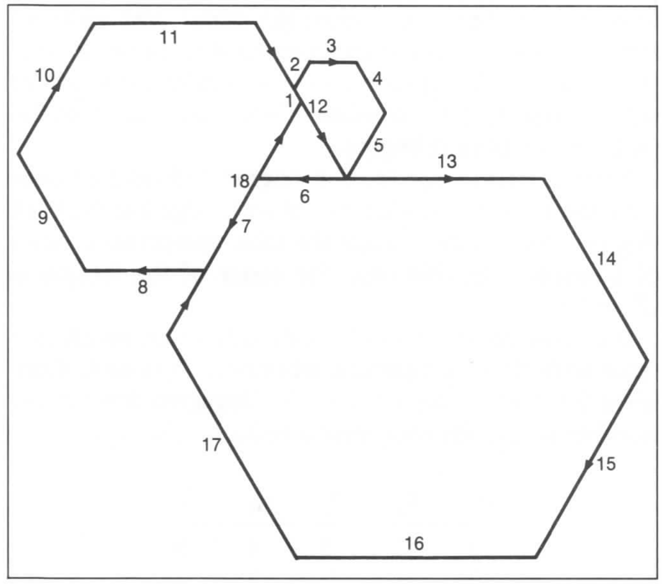


Figure 11. An isogon of order 18, with $\alpha = 120°$. The turning angle is 60° or 1/6 of 360°, so that D, the number of edge directions (6) is of form $D = 4k + 2$, $k = 1$.

In our examples for even D, τ was respectively 1/8 and 1/6 of 360 degrees. Cases for which the numerator is greater than 1 are analogous to that looked at for odd D, as in Figure 12, where τ is 3/8 of 360°.

This completes our survey of Cornet's proof that a serial isogon is always constructible for any angle that is a rational fraction of 360 degrees. Paths generated by his rule are frequently ornamental flowers (or fireworks?) as seen in Figure 13.

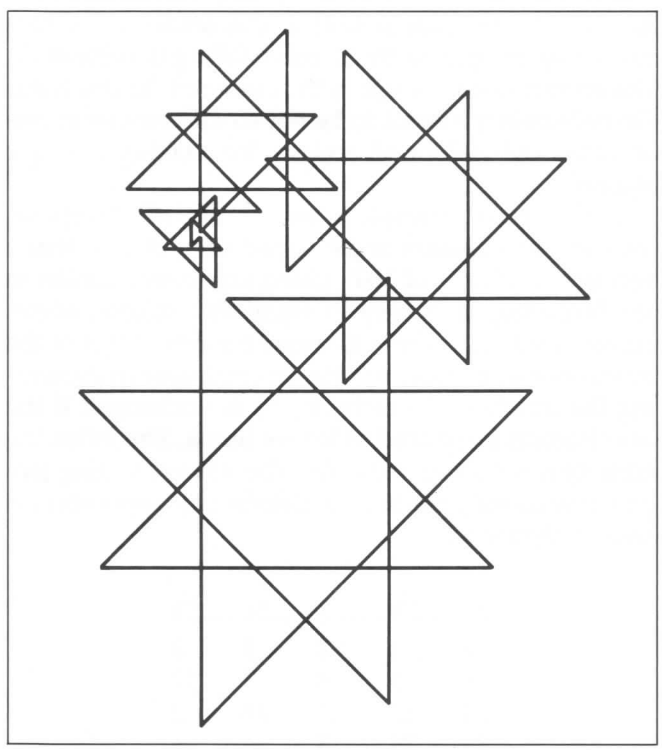


Figure 12. A path of order 32, $\alpha = 45°$. This is a relative of Figure 9; τ is now 3/8 of 360°. Cornet's rule is unaffected by the change in numerator.

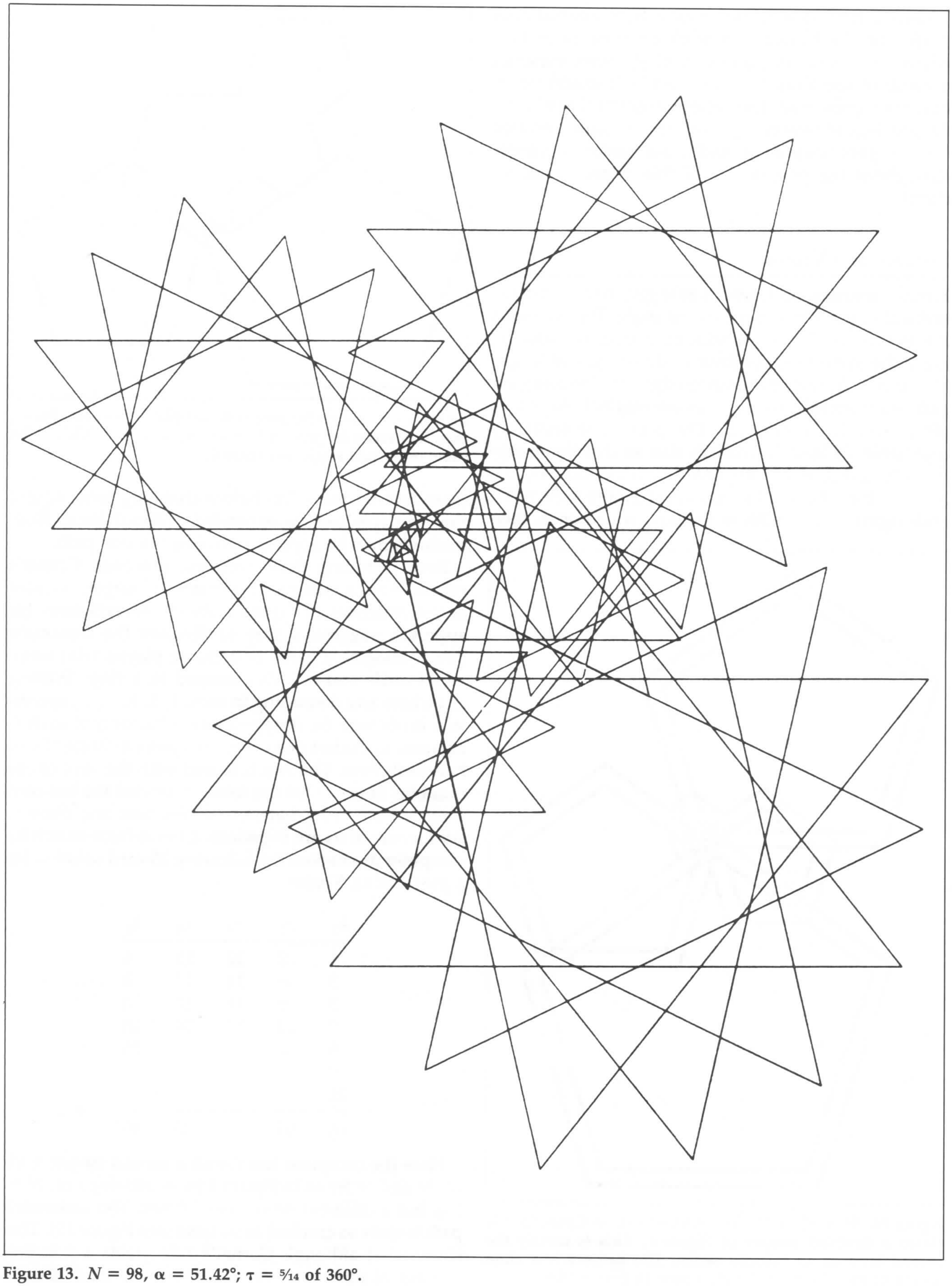


Figure 13. $N = 98$, $\alpha = 51.42°$; $\tau = 5/14$ of 360°.

Hans Cornet is a retired high school mathematics teacher in The Hague. His work on serial isogons—which continues—is pursued entirely from personal interest; in the absence of this article it would never have been published. I am sincerely grateful to him for the privilege of presenting a significant result, and also for his generous help and kind encouragement throughout the preparation of this paper. Dank je, Hans!

Patience is a Virtue

Trivial variations on Cornet's rule give rise to endless series of isogons for every rational angle. Two methods are as follows. 1) On completing a path of order N, using the same rule, continue adding edges of length $N + 1$, $N + 2$, and so on, up to edge $2N$. The resulting path is a "second harmonic" of the original. An example is seen in Figure 3(d). The process is endlessly repeatable. 2) Modify Cornet's rule so that the exceptional turns fall on $2D$, $4D$, $6D$, etc. (rather than D, $2D$, $3D$, . . .); see Figure 3(e). Compare Figure 6 ($N = 25$) with Figure 14 ($N = 50$) to see how we go once, then twice around the spiral before changing tack. Again, the principle can be expanded without limit. Both methods may be applied alternately in one path.

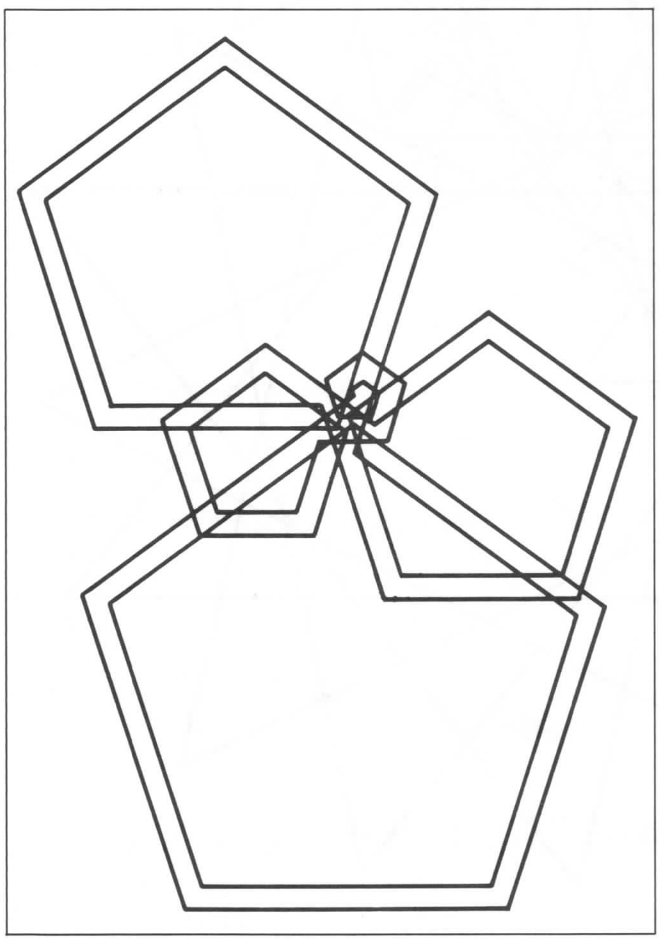


Figure 14. $N = 50$, $\alpha = 108°$. A variation on Cornet's rule yields a doubled version of Figure 6. This is merely the second term of an infinite series. The process is a close cousin of the "harmonic" effect seen in Figure 3(d).

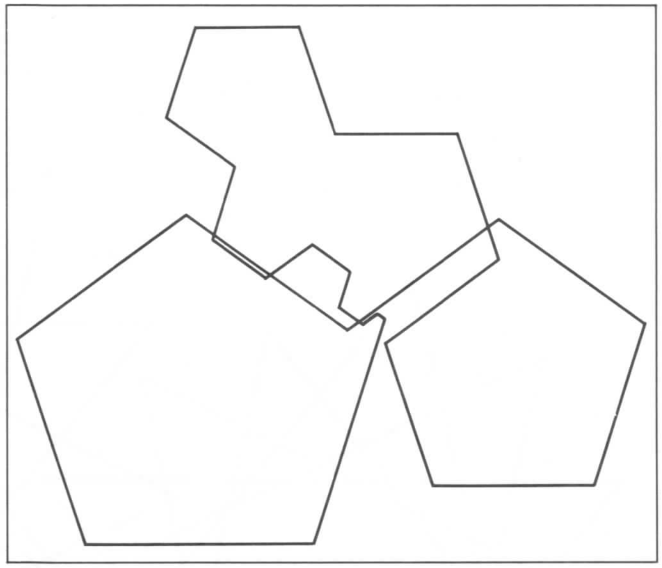


Figure 15. A computer program that plays patience discovered this serial isogon for $N = 25$ and $\alpha = 108°$. This is the same order and angle as Figure 4.

Even including these variants, however, Cornet's rule does not account for every rational isogon, as most of the examples of Figure 3 attest. Nevertheless, his approach suggests a way to discover the remaining paths. Imagine a game of patience played with numbered cards and H hats arranged in a ring. Starting anywhere and discarding in turn, 1, 2, 3, . . . , successive cards may be dropped into a hat only if each is adjacent to the last one used. This gives a choice of two hats each time. The aim is to end with the sum of the numbers in every hat the same, provided the last card thrown occupies a hat next to the first one chosen. Alas, some of us are *im*patient; a brute-force search by computer discovered the following 25-card solution for a game using 5 hats:

h_1	h_2	h_3	h_4	h_5
1	2	12	11	4
3	6	14	13	8
5	15	16	17	10
7	20	23	24	18
9	22			25
19				
21				
65	65	65	65	65

Here the computer has found a second isogon with angle and order as in Figure 6 ($\alpha = 108$ degrees, $N = 25$), but a different sequence of turns. The associated path is quite as crooked as its table (see Figure 15). This shows that although Cornet's rule yields a winning strategy at patience, other solutions may exist. Except in a few instances for small N, however, the number of solutions extant for each angle/order cannot, as yet, be predicted. As it happens, Figure 15 is one of three distinct solutions for this angle and order. Some readers may enjoy trying to find the missing one for themselves.

The isogon in Figure 6 has a further remarkable feature. Recall that 108 degrees is the inside angle of a regular pentagon. Now look closely at the area bounded by edges 5, 10, 15, 20 and 25, in the centre of the figure. It can be proved that the shape outlined there is indeed a regular pentagon!

After this, readers may not be surprised to learn that a regular heptagon nestles at the centre of the analogous path using the heptagon angle of 128.57 degrees. Surprisingly, however, there the pattern ends, for no other such polygons have been found in comparable paths for different angles. This curiosity remains to be explained.

Irrational Isogons

Not every isogon is detectable in the way described above. Figure 16 shows a 6-sided path using an angle not expressible as a rational fraction of 360 degrees. Here $\alpha = \arccos(3/4)$ radians ($= 41.4096 \ldots$ degrees), as the added parallelogram construction serves to illustrate. Moreover, this is in fact the smallest (shortest path) serial-sided isogon of all. Its discovery is due to a computer program that uses a turtle approach to plot paths as it executes a brute-force search for isogons of any order. Even for N as small as 6, however, the number of different possible angles between 0 and 180 degrees remains infinite. What kind of a program can examine paths for them *all*? My eventual algorithm turned out surprisingly simple, although human-assisted.

In the program, after specifying some N and α, the order and angle of a path to be investigated, simple trigonometry is used to determine the positions of successive vertices for every possible path, one after another. A path is just a sequence of left/right turns, represented as a string of N bits: 0 = left, 1 = right. When called, a standard routine loads an array with a new permutation of bits that now defines the next path to be plotted. Starting at the origin of the Cartesian plane, edge 1 of the path is assumed coincident with the positive x-axis. On completing a path, the coordinates of the end point of the final segment can be checked. If these were again 0,0 then we would have a closed path, and if the angle calculated between the final segment and the x-axis was again α, then the sequence of turns under test would give rise to a serial isogon.

What I did was to accept any test whose end point lay within a small window centered on the origin, and whose reentering angle was within 2° of α; then I

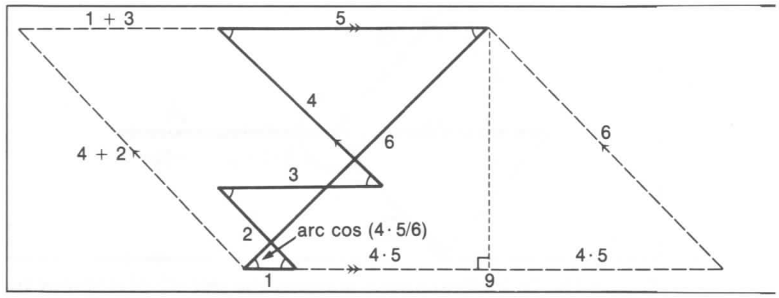


Figure 16. The smallest serial isogon of all is an order 6 path using an angle that cannot be expressed as a rational fraction of 360°: arc cos($\frac{3}{4}$). Segments lie along 3 directions.

retested with slightly incremented or decremented path angle to home in on an apparent solution.

As the last stage, pencil-and-paper work is necessary to make mathematical sense of the angle empirically arrived at, and verify it is really a solution. For instance, 41.4096 . . . degrees means little until independent reasoning reveals it as arc cos($\frac{3}{4}$) radians, as in Figure 16.

In practice, running time on my PC became prohibitive for orders above 16. This could doubtless be improved upon, if desired, although examination of higher N is still within reach if the search is restricted to a single angle. In the latter case, when the angle is irrational, a little thought shows that testing paths of uneven order can be skipped.

Beyond order 6, the next largest isogons brought to light by the program are two of order 8: one of them the tiling polyomino, the other a related path, but again using an irrational angle (see Figure 17). This fresh discovery prompted a new result covering irrational isogons in which just 3 edge directions, the minimum possible, occur. Below we shall prove that in that case either $N = 8k$ and $\alpha = \arccos(4k/(4k + 1))$, or $N = 8k - 2$ and $\alpha = \arccos((4k - 1)/4k)$, where k is an integer.

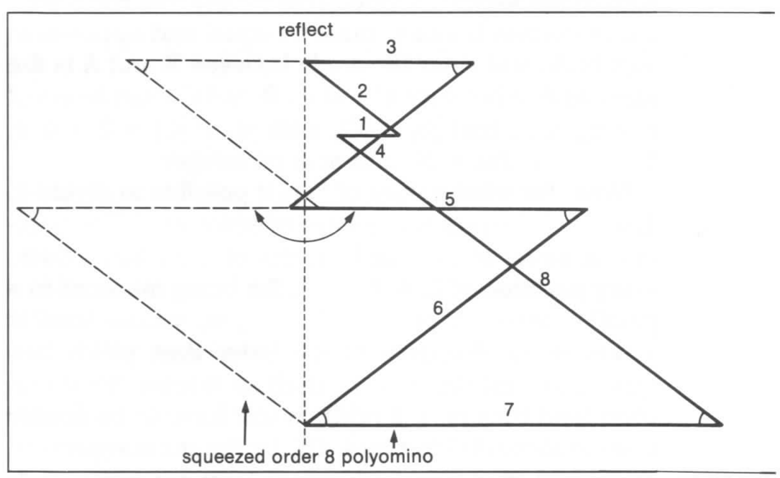


Figure 17. There are two serial isogons of order 8: the tiling polyomino and this related path using an irrational angle of arc cos ($\frac{4}{5}$). The figure illustrates their relationship. Segments again exhibit 3 directions.

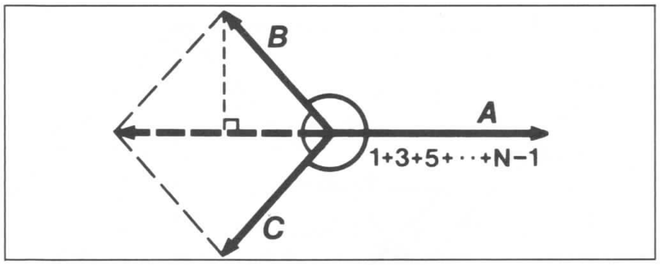


Figure 18. In tridirectional isogons the net displacement in each direction can be represented by three vectors.

Irrational Isogons Using 3 Directions

Consider a tridirectional isogon in which α (and thus τ) is irrational. The net displacement away from the path origin in each direction may be represented by 3 vectors, **A, B, C,** the angles between them being the same as those between their corresponding directions, as in Figure 18. This will be τ in two out of three cases only, for otherwise 3τ would equal 360°, implying a rational τ. Evidently the remaining angle is $2 \cdot 180 - 2\tau = 2\alpha$. Thus only one of the three vectors is "central" in bisecting the angle between the two others: **A.** Significant inferences now follow from this.

Recall that adjacent isogon edges can only occupy directions separated by angle τ. Suppose the direction of edge 1 is that of **A**, the central vector (later we shall see that this must be the case). Then that of edge 2 is **B** or **C**. But the angle between **B** and **C** is not τ, and so edge 3 can only belong again to **A**. Similar logic applies to succeeding cases, showing that edges of uneven length must all point in direction **A**, whose magnitude, $|\mathbf{A}|$, is thus $1 + 3 + 5 + \ldots + (N - 1)$ (recall N is even). Hence $|\mathbf{B}| + |\mathbf{C}|$, the combined lengths of the residual edges, must equal the sum of the remaining even numbers.

Moreover, since the path represented is closed, the sum of the vectors is zero. This means that the resultant of vectors **B** and **C** must be equal and opposite in sign to **A**, and since the angle between **B** and **A** is the same as that between **C** and **A**, **B** and **C** must be equal in length, so that $|\mathbf{B}| = |\mathbf{C}|$, with $|\mathbf{B}| + |\mathbf{C}| = 2 + 4 + 6 + \ldots + 2m = N$, and m is an integer.

Now, for what values of N is it possible to divide 2, 4, 6, . . . , $2m$ into two groups of equal sum? The question is easier to answer in terms of their half values, every partition of 2, 4, 6, . . . , $2m$ being mirrored in a parallel partition of 1, 2, 3, . . . , m, whose total is $\frac{1}{2}m(m + 1)$. Bisection of the latter then yields two groups of sum $\frac{1}{4}m(m + 1)$, itself an integer. So, if m is even (and thus $m + 1$ odd), m will have to be doubly even to allow division by 4. Or, by the same argument, if m is odd, then the doubly even term must be $m + 1$. In summary, either $m = 4k$ or $m = 4k - 1$, where k is an integer. However, $2m = N$, which shows that $N = 8k$ or $N = 8k - 2$.

Then, as the perpendicular in the vector diagram helps to show:

$$\begin{aligned} |\mathbf{B} + \mathbf{C}| &= 2 \cdot |\mathbf{B}| \cdot \cos(\alpha) \\ &= 2 \cdot |\mathbf{C}| \cdot \cos(\alpha) \\ &= (|\mathbf{B}| + |\mathbf{C}|) \cdot \cos(\alpha) \\ &= |\mathbf{A}|, \text{ as seen above,} \end{aligned}$$

so that

$$\cos(\alpha) = \frac{|\mathbf{A}|}{|\mathbf{B}| + |\mathbf{C}|} = \frac{1 + 3 + 5 + \ldots + N - 1}{2 + 4 + 6 + \ldots + N} = \frac{N^2/4}{(N + 2)N/4} = \frac{N}{N + 2}$$

from which $\alpha = \arccos(N/(N + 2))$;

or $\alpha = \arccos(4k/(4k + 1))$ when $N = 8k$

and $\alpha = \arccos((4k - 1)/4k)$ when $N = 8k - 2$,

which is what we set out to prove.

We have yet to see what happens if edge 1 is aligned with a non-central vector, **B** or **C**. It is easy to see that

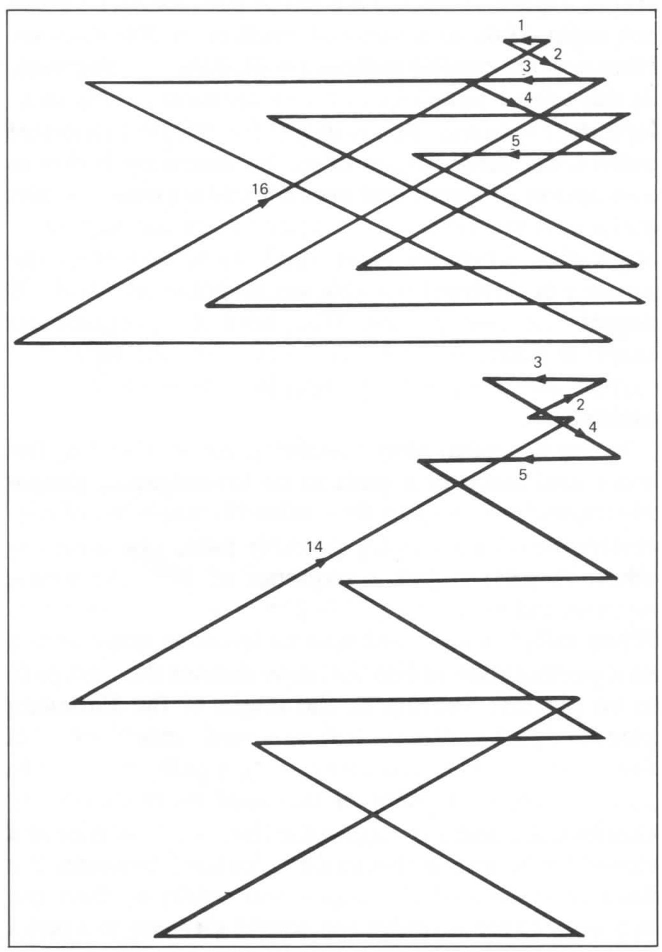


Figure 19. Tridirectional paths using irrational angles have orders of form $N = 8k$ and $8k - 2$; here $k = 2$. Above: one of the 7 paths of order 16; $\alpha = \arccos(8/9)$. Below: one of the 4 paths of order 14; $\alpha = \arccos(7/8)$.

the roles are then reversed, with $|\mathbf{A}| = 2 + 4 + 6 + \ldots + N$ and $|\mathbf{B}| + |\mathbf{C}| = 1 + 3 + 5 + \ldots + (N - 1)$. However, since the first total is greater than the second, a zero vector sum would be impossible, even if τ were 180°. Hence, no such path exists.

Figures 16 and 17 show the single instances of such isogons for $k = 1$; what happens beyond? For $k = 2$ the computer finds four of order 14 and seven of order 16, these two totals reflecting the number of distinct partitions of 2, 4, . . . , 14 and 2, 4, . . . 16 into two subsets of equal sum: the edges assigned to directions **B** and **C**; see Figure 18. More generally, a partition scheme yielding solutions for every k is as follows:

$N = 8k$:
{1}: 2,6,10, . . . , $4k - 2$; $4k + 4$, $4k + 8$, $4k + 10$, . . . , $8k$
{2}: 4,8,12, . . . , $4k - 4$; $4k$,$4k + 2$, $4k + 6$, . . . , $8k - 2$

$N = 8k - 2$:
{1}: 2,6,10, . . . , $4k - 2$; $4k$,$4k + 4$, $4k + 8$, . . . , $8k - 4$
{2}: 4,8,12, . . . , $4k - 4$; $4k + 2$,$4k + 6$,$4k + 10$, . . . , $8k - 2$

The above scheme is easily verified by summing the component arithmetic series and comparing partition totals. In cases of 3 directions, therefore, we are able to create and count paths for every possible order.

As might be expected, not every irrational isogon discovered by computer is tridirectional. Research into these more complicated types continues.

Pretty Polyiamonds

Variations on the serial-sided theme will have occurred to readers: paths whose edge-lengths listen to different laws: arithmetic or geometric series, sequences of primes, etc. Paths in higher dimensions also await investigation. To conclude this survey of serial planar types, however, I would like to mention one further variation.

I already reported the disappointment that no more serial isogons have been found that tile. Figure 1 remains unique in this respect. But Figure 1 is a polyomino: a figure on a right-angled grid that can itself be tiled with squares. In this light, another avenue to ex-

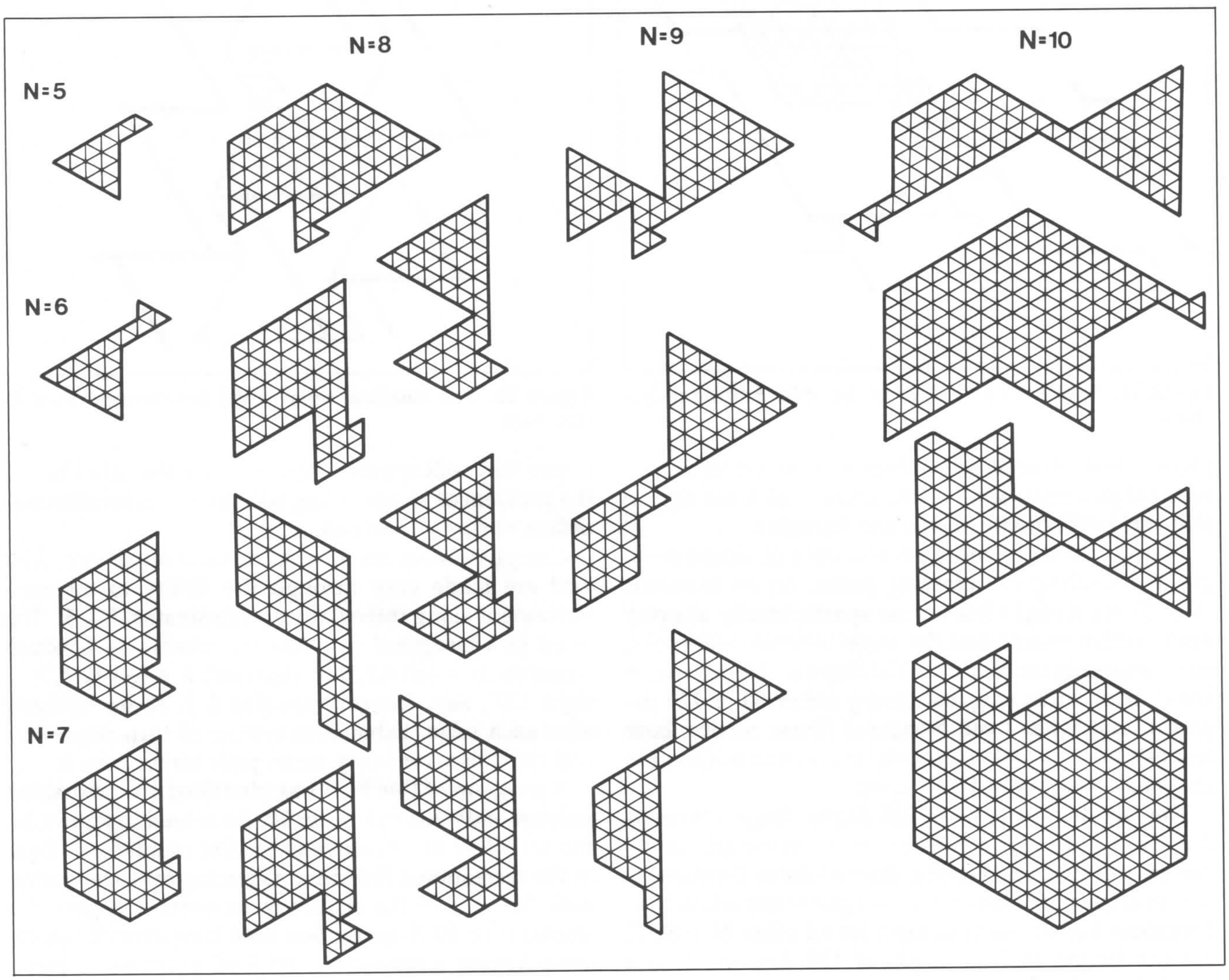


Figure 20. The serial-sided polyiamonds of order $N \leq 10$.

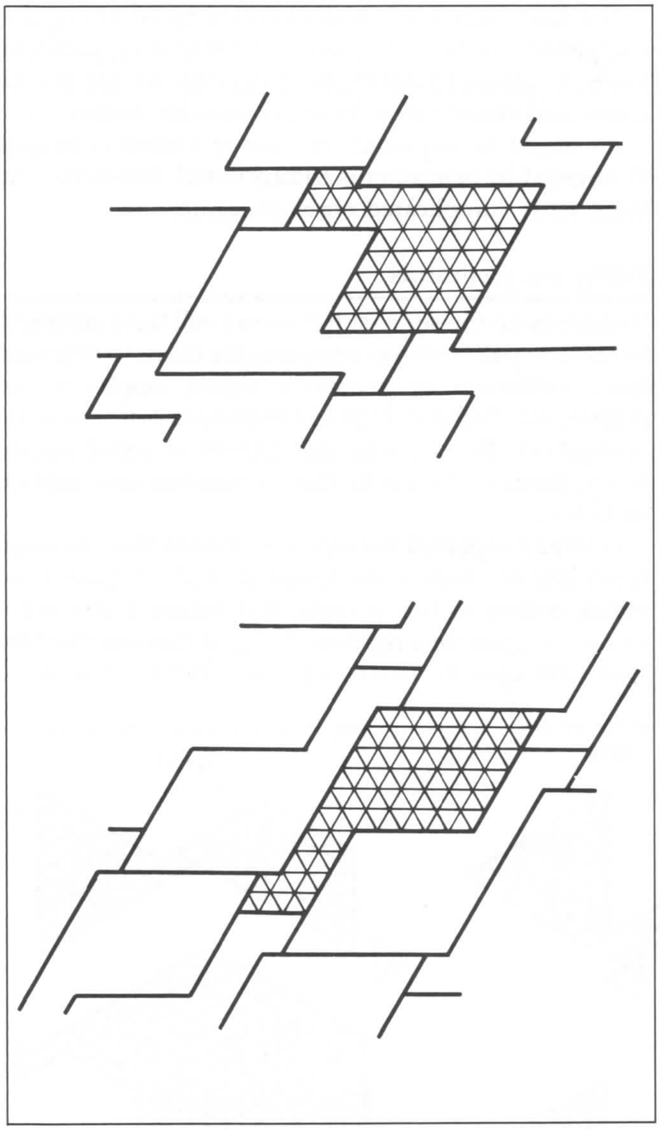


Figure 21. Polyiamond analogs of the order 8 tiling polyomino.

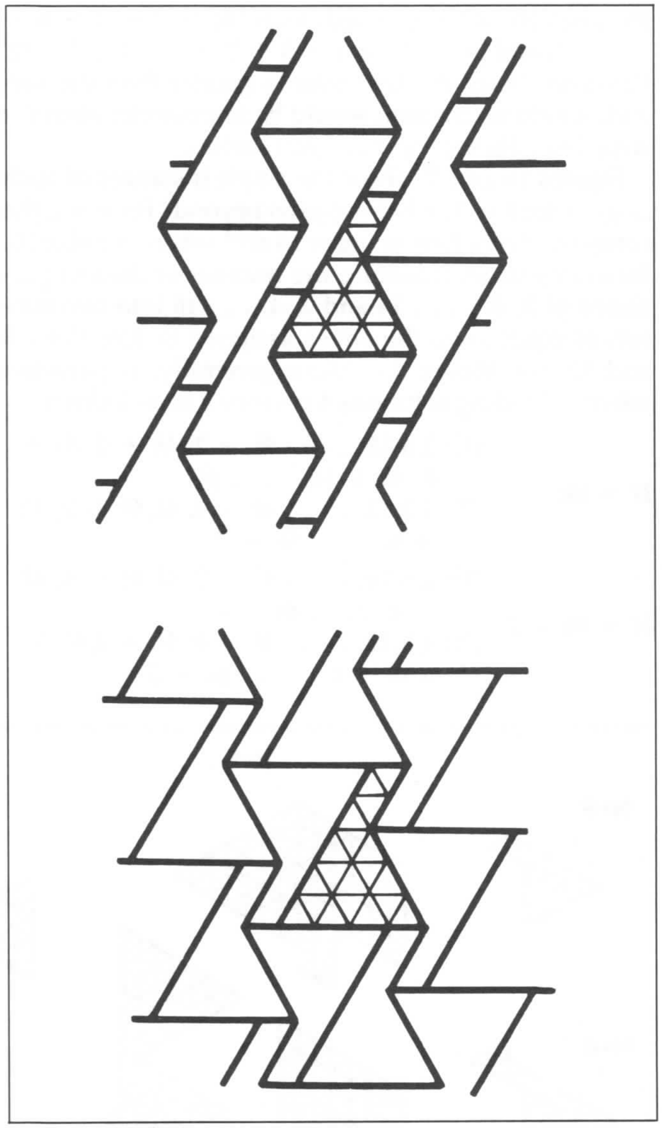


Figure 22. The smallest serial-sided polyiamond tiles in two ways.

plore is that of serial-sided shapes on an *isometric* grid, or in other words, *serial polyiamonds*, which are figures that can be tiled with equilateral triangles.

Consider first the more general case of closed serial paths, including self-crossing paths, on an isometric grid. Three straight lines cross symmetrically at every node, which means that the angle between successive path segments can be 60 or 120 degrees. Hence, paths are of two kinds: isogonal (using either 60 or 120 degrees), and what I call *bi*sogonal (those mixing both angles). Then, serial-sided polyiamonds correspond to the *simple polygons* of both types.

The fact that every 60 or 120 degree isogon (such as those in Figure 3) is a path on an isometric grid forms the basis of two neat results, due to Martin Gardner: 1) For an angle of 60 degrees, no isogon exists when $N = 1$ modulo 3 (they seem to exist for all other $N > 8$); 2) for any isogon with an angle of 120 degrees, N is a multiple of 6. The proofs are not very difficult; one hopes they will appear elsewhere. On the other hand, the problem of enumerating 60°/120° paths for different orders remains unsolved.

Computer searches for closed paths on an isometric grid are made easy through the ability to measure movement along three (lattice) coordinates, *I*, *J*, *K*. The turns in a bisogonal path are encodeable as 4-valued elements: 0 = left 60°, 1 = right 60°, 2 = left 120°, 3 = right 120°, say. Integer variables *I*, *J*, *K* are updated after each edge. Exhaustive testing of turn sequences will thus discover every serial path for a given *N*.

A program of this kind has identified 18 serial-sided polyiamonds through order 10. Presented in Figure 20, the set offers an attractive extension to a familar topic in the recreational literature. Glancing over the group, note that one of the shapes is an order-9 isogon, the smallest for 60 degrees. See next how two of the order-8 figures resemble the original polyomino. However, a moment's thought shows that square grid

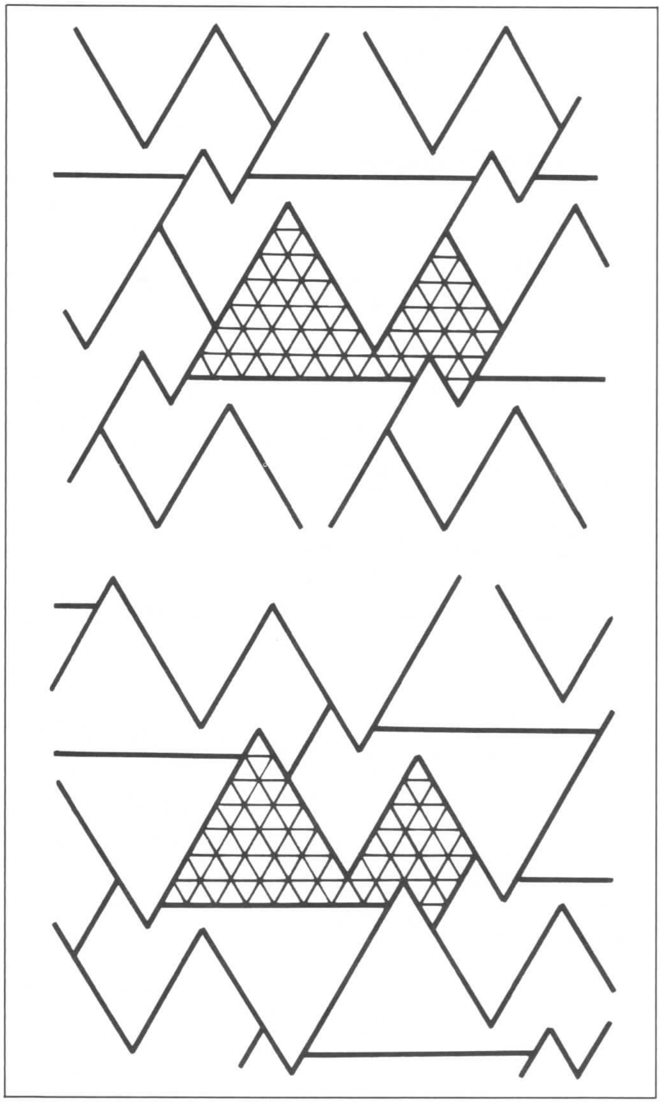


Figure 23. Two tilings by a serial-sided polyiamond of order 8.

paths can always be projected onto a parallelogram lattice, after which it comes as no surprise that these shapes tile analogously to the former (Figure 21). More pleasing is the presence of two genuinely new pretty polyiamonds, each of which paves in two different ways (Figure 22 and 23). This soon leads into the realm of serial-sided *tiles,* in general. But that is another message in a different bottle from yet a further ocean.

References

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